Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Primer Vector Theory for Matched-Conic Trajectories

DAVID R. GLANDORF*
Lockheed Electronics Company, Houston, Texas

Nomenclature

H = Hamiltonian
J = performance index
k = gravitational parameter
P^* = position vector of a planet at time t^*
Q^* = velocity vector of a planet at time t^*
R = position vector of the spacecraft
r = magnitude of R
r_s = sphere of influence radius of a planet
t = time
t^* = time at which the vehicle crosses a sphere of influence
V = velocity vector of the spacecraft
$\gamma,\lambda,\mu,\nu,\sigma$ = Lagrange multipliers
Φ = state transition matrix
ψ = over-all transition matrix for λ and $\dot{\lambda}$

Subscripts

0 = evaluation at the initial time
f = evaluation at the final time
$-$ = evaluation just before t^*
$+$ = evaluation just after t^*

Superscript

T = transpose

Introduction

PRIMER vector theory as introduced by Lawden,[1] extended to nonoptimal trajectories by Lion and Handelsman,[2] and applied by Jezewski and Rozendaal,[3] has become an effective method for determining optimal n-impulse trajectories for motion in an inverse square law central gravitational field. The theory is further extended here for application to matched-conic trajectories. The primary result of this analysis is that the primer vector is continuous at a sphere of influence whereas its time derivative is discontinuous. It is also shown that the method of finding optimal n-impulse trajectories presented in Ref. 3 is readily extended to matched-conic trajectories.

Matched-Conic Trajectories

If the vehicle is within the sphere of influence of any planet, it is in a planetocentric phase, and its motion is considered to be governed only by the gravitational field of the planet. If the vehicle is outside of the spheres of influence of all planets, it is in a heliocentric phase, and only the gravitational field of the sun is considered in determining its motion. Since all gravitational fields are assumed to be of the inverse square law form, the trajectory consists of a series of conic arcs described by the solution to the equations

$$\dot{R} - V = 0,\ \dot{V} + (k/r^3)R = 0 \tag{1}$$

where R and V are measured relative to the appropriate central body that has a gravitational parameter k.

An inertial coordinate system is used during a heliocentric phase. During a planetocentric phase, the reference coordinate system is centered on the moving planet and is parallel to the heliocentric reference frame. Although the position and velocity are inertially continuous throughout the trajectory, the translation of coordinate systems when the vehicle passes through a sphere of influence causes a discontinuity in the state vector given by

$$R_+ = R_- \pm P^*,\ V_+ = V_- \pm Q^* \tag{2}$$

where the $+$ is used if the vehicle is leaving a sphere of influence and the $-$ is used if the vehicle is entering a sphere of influence.

Adjoint Equations

Consider a two-impulse, fixed-time, matched-conic trajectory between specified initial and final position vectors which consists of a planetocentric phase followed by a heliocentric phase. The time at which the vehicle crosses the sphere of influence is implicitly defined by the equation

$$R_-^T R_- - r_s^2 = 0 \tag{3}$$

The performance index J is defined to be the sum of the magnitudes of all impulses. Thus, for a two-impulse trajectory,

$$J = |V_{0+} - V_{0-}| + |V_{f+} - V_{f-}| = |\Delta V_0| + |\Delta V_f| \tag{4}$$

Following the approach of Bryson et al.,[4] Eqs. (1–3) are adjoined to the performance index to give

$$I = J + \nu^T(R_+ - R_- - P^*) + \mu^T(V_+ - V_- - Q^*) + \gamma(R_-^T R_- - r_s^2) + \int_{t_0}^{t_-^*} + \int_{t_+^*}^{t_f} \left[\sigma^T(\dot{R} - V) + \lambda^T\left(\dot{V} + \frac{k}{r^3}R\right)\right] dt$$

Formation of the differential of I gives

$$dI = \left(\lambda_f - \frac{\Delta V_f}{|\Delta V_f|}\right)^T dV_{f-} - \left(\lambda_0 - \frac{\Delta V_0}{|\Delta V_0|}\right)^T dV_{0+} + (\sigma_- - \nu + 2\gamma R_-)^T dR_- + (\nu - \sigma_+)^T dR_+ + (\lambda_- - \mu)^T dV_- + (\mu - \lambda_+)^T dV_+ + (\lambda_+^T \dot{V}_+ + \sigma_+^T V_+ - \lambda_-^T \dot{V}_- - \sigma_-^T V_- - \mu^T \dot{Q}^* - \nu^T \dot{P}^*)dt^* + \int_{t_0}^{t_-^*} + \int_{t_+^*}^{t} \left\{(-\sigma - \dot{\lambda})^T \delta V + \left[\frac{k}{r^3}\lambda - \frac{3k}{r^5}(\lambda^T R)R - \dot{\sigma}\right]^T \delta R\right\} dt$$

For I to be stationary under arbitrary perturbations, the coefficients of dV_{f-}, dV_{0+}, dR_-, dR_+, dV_-, dV_+, dt^*, δV, and δR must vanish. The following results are thus obtained:

$$\lambda_0 = \Delta V_0/|\Delta V_0|,\ \lambda_f = \Delta V_f/|\Delta V_f| \tag{5}$$

$$\lambda_+ = \lambda_- = \mu \tag{6}$$

Received April 28, 1969. This work was supported by NASA, Manned Spacecraft Center, Houston, Texas under contract NAS9-5384.

* Staff Engineer, Flight Mechanics Section. Member AIAA.

$$\sigma_+ = \sigma_- + 2\gamma R_- = \nu \quad (7)$$

$$\lambda_+^T \dot{V}_+ + \sigma_+^T V_+ = \lambda_-^T \dot{V}_- + \sigma_-^T V_- + \mu^T \dot{Q}^* + \nu^T \dot{P}^* \quad (8)$$

$$\dot{\lambda} = -\sigma \quad (9)$$

$$\dot{\sigma} = (k/r^3)\lambda - (3k/r^5)(\lambda^T R)R \quad (10)$$

Equations (9) and (10) can be combined to give the standard second-order equation for the primer vector λ;

$$\ddot{\lambda} = -(k/r^3)\lambda + (3k/r^5)(\lambda^T R)R \quad (11)$$

Manipulation of Eqs. (6–9) provides the following results:

$$\dot{\lambda}_+ = \dot{\lambda}_- - 2\gamma R_- \quad (12)$$

$$2\gamma = \lambda_-^T(\dot{V}_- - \dot{V}_+ + \dot{Q}^*)/R_-^T V_- \quad (13)$$

$$H_+ = H_- + \lambda_+^T \dot{Q}^* - \dot{\lambda}_+^T Q^* \quad (14)$$

where H is the Hamiltonian defined by

$$H = \lambda^T \dot{V} - \dot{\lambda}^T V \quad (15)$$

Equations (12) and (13) can be combined to give

$$\dot{\lambda}_+ = \dot{\lambda}_- + \rho\zeta^T \lambda_- \quad (16)$$

where

$$\rho = R_-/R_-^T V_-,\ \zeta = \dot{V}_+ - \dot{V}_- - \dot{Q}^*$$

Equations 6 and 16 can now be written in matrix form to give

$$\begin{bmatrix} \lambda \\ \dot{\lambda} \end{bmatrix}_+ = \begin{bmatrix} I & 0 \\ \rho\zeta^T & I \end{bmatrix} \begin{bmatrix} \lambda \\ \dot{\lambda} \end{bmatrix}_- \quad (17)$$

Equations (6, 14, and 16) show that the primer vector is continuous across a sphere of influence whereas its time derivative and the Hamiltonian are discontinuous.

Analysis similar to that given previously for the case of a two-impulse trajectory, consisting of a heliocentric phase followed by a planetocentric phase, shows that Eq. (14) becomes

$$H_+ = H_- - \lambda_-^T \dot{Q}^* + \dot{\lambda}_-^T Q^* \quad (18)$$

and that Eq. (17) is still valid provided that ρ and ζ are redefined as

$$\rho = R_+/R_+^T V_+,\ \zeta = \dot{V}_+ - \dot{V}_- + \dot{Q}^*$$

Obtaining Optimal Solutions

The method presented in Ref. 3 for finding optimal n-impulse trajectories in a single, inverse square law, central gravitational field is based on the equation

$$\lambda^T \delta V - \dot{\lambda}^T \delta R = \text{const} \quad (19)$$

It is now shown that Eq. (19) is satisfied across a sphere of influence and thus along the entire matched-conic trajectory.

At a sphere of influence, the noncontemporaneous variations are given by

$$\begin{aligned} dR_- &= \delta R_- + V_- dt^*,\ dV_- = \delta V_- + \dot{V}_- dt^* \\ dR_+ &= \delta R_+ + V_+ dt^*,\ dV_+ = \delta V_+ + \dot{V}_+ dt^* \end{aligned} \quad (20)$$

Substitution of Eqs. (2) into Eqs. (20) and rearrangement gives

$$\delta R_+ = \delta R_-,\ \delta V_+ = \delta V_- - \zeta\, dt^* \quad (21)$$

An expression for dt^* can be formed as follows. The total variation in the radius magnitude at the sphere of influence must vanish so that

$$dr_- = \delta r_- + \dot{r}_-\, dt^* = 0$$

or

$$dt^* = -\delta r_-/\dot{r}_- \quad (22)$$

But since $r_-^2 = R_-^T R_-$, it follows that $\delta r_- = R_-^T \delta R_-/r_-$ and $\dot{r}_- = R_-^T V_-/r_-$, so that Eq. (22) becomes

$$dt^* = -R_-^T \delta R_-/R_-^T V_- = -\rho^T \delta R_- \quad (23)$$

and Eqs. (21) become, in matrix form,

$$\begin{bmatrix} \delta R \\ \delta V \end{bmatrix}_+ = \begin{bmatrix} I & 0 \\ \zeta\rho^T & I \end{bmatrix} \begin{bmatrix} \delta R \\ \delta V \end{bmatrix}_- \quad (24)$$

Equations (17) and (24) can now be combined to give

$$\begin{aligned} \lambda_+^T \delta V_+ - \dot{\lambda}_+^T \delta R_+ &= \lambda_-^T(\zeta\rho^T \delta R_- + \delta V_-) - \\ &\qquad (\rho\zeta^T \lambda_- + \dot{\lambda}_-)^T \delta R_- \\ &= \lambda_-^T \delta V_- - \dot{\lambda}_-^T \delta R_- \end{aligned}$$

so that Eq. (19) is satisfied across a sphere of influence.

For the two-impulse reference trajectory considered in the development of the adjoint equations, the initial and final values of the primer vector and its derivative are related by

$$\begin{bmatrix} \lambda \\ \dot{\lambda} \end{bmatrix}_{t_f} = \psi(t_f,t_0) \begin{bmatrix} \lambda \\ \dot{\lambda} \end{bmatrix}_{t_0} \quad (25)$$

where

$$\psi(t_f,t_0) = \Phi(t_f,t^*)W(t^*)\Phi(t^*,t_0) \quad (26)$$

and

$$W(t^*) = \begin{bmatrix} I & 0 \\ \rho\zeta^T & I \end{bmatrix} \quad (27)$$

The analysis of Ref. 2 therefore is readily extended to matched-conic trajectories provided that Eq. (17) is used to propagate the primer vector and its derivative across a sphere of influence, and the over-all transition matrix ψ is used to compute the initial value of the time derivative of the primer vector for each two-impulse subarc of the trajectory. A method of computing the state transition matrix for each phase of a matched-conic trajectory can be found in Ref. 5.

Summary

Primer vector theory, which has been effectively used to compute optimal n-impulse trajectories in an inverse square law, central gravitational field, has been extended for application to matched-conic trajectories. The primer vector has been shown to be continuous at a sphere of influence whereas its time derivative and the Hamiltonian are discontinuous. The discontinuities, however, are not arbitrary, but can be explicitly calculated.

Although the trajectories considered in the analysis consisted of only two phases, the results are applicable to a trajectory that crosses any number of spheres of influence.

References

[1] Lawden, D. F., *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963, pp. 77–81.

[2] Lion, P. M. and Handelsman, M., "Primer Vector on Fixed-Time Impulsive Trajectories," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, pp. 127–132.

[3] Jezewski, D. J. and Rozendaal, H. L., "An Efficient Method for Calculating Optimal Free-Space N-Impulse Trajectories," *AIAA Journal*, Vol. 6, No. 11, Nov. 1968, pp. 2160–2165.

[4] Bryson, A. E., Denham, W. F., and Dreyfus, S. E., "Optimal Programming Problems with Inequality Constraints. I: Necessary Conditions for Extremal Solutions," *AIAA Journal*, Vol. 1, No. 11, Nov. 1963, pp. 2544–2550.

[5] Glandorf, D. R., "Lagrange Multipliers and the State Transition Matrix for Coasting Arcs," *AIAA Journal*, Vol. 7, No. 2, Feb. 1969, pp. 363–365.